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Role of vertical temperature gradient in damping/amplification of atmospheric waves

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Abstract : Considering an axial symmetric flow pattern in a non-homogeneous and non-uniform temperature region, the analytical conditions for damping/amplification of the vertically propagating waves parallel to the axis of symmetry have been obtained.

The analysis has been applied to the different regions of Earth's lower and middle atmosphere. Different limits of frequency have been identified for damping/amplification of the waves in different regions. It has been found that in a region of positive vertical temperature gradient (Stratosphere) the waves may undergo damping, while in a region of negative temperature gradient (Mesosphere or Troposphere) the said waves may undergo amplification provided the wave frequency lies in a specific range.

Keywords : Vertical temperature gradient, wave frequency, damped and amplified waves, Earth's atmosphere

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1. Introduction

The dynamical response of heating in the upper atmosphere has been investigated by many researchers [1-3]. Even for astrophysical plasma, heating has been observed due to damping of the hydromagnetic waves [4]. Investigations have also been done for the heating and cooling in the lower and middle atmosphere of the Earth. Matsuno [5] suggested a sudden warming in the stratosphere when a meso-scale disturbance in the troposphere propagates upward, and this warming in the stratosphere, in turn, is accompanied by equally strong mesospheric cooling [6]. Observational evidence of mesospheric cooling has also been noticed [7] and the damping rate of temperature disturbances in the mesosphere has been derived [8].

Now the heating and cooling in a medium are, most possibly, associated with damping and amplification of the waves. With this in mind, an attempt has been made to investigate how the vertical temperature gradient can influence the damping or amplification of a vertically propagating wave through the different regions of opposite temperature gradient in the lower and middle atmosphere. Using the characteristic equation (with the help of the perturbation technique) for the vertically propagating waves, it has been shown analytically and numerically that in a region of positive temperature gradient in the Earth's atmosphere, the waves may undergo damping, while in a region of negative temperature gradient they may undergo amplification provided the wave frequency lies in a specific range.

2. Basic equations, assumptions and dispersion relation

The set of equations considered to describe the motion of a general hydrodynamic wave with particular reference to the Earth's atmosphere are as follows :

a) the equation of motion

$$\frac{dV}{dt} = -\frac{1}{\rho}\nabla p - 2[\boldsymbol{\Omega} \times V] + \frac{1}{\rho}F \; ; \tag{2.1}$$

b) the eqn.of continuity for a compressible fluid

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho V) = 0 \; ; \tag{2.2}$$

c) the thermodynamic eqn. for an adiabatic system

$$C_p \frac{dT}{dt} - \frac{1}{\rho}\frac{dp}{dt} = 0; \quad \text{and} \tag{2.3}$$

d) the equation of state

$$P = \rho RT, \tag{2.4}$$

where V is the velocity vector, p is the pressure, ρ is the density, Ω is the angular velocity of the Earth, F is the frictional force/unit volume and the other terms have their usual meaning.

Now, since many atmospheric disturbances in meso-scale and synoptic-scale possess a structure which is vertically spiralling, the above equations have been transformed in cylindrical polar co-ordinates (r,θ,z). The axis of the cylindrical frame representing the atmospheric disturbance has been taken as normal to the plane tangential to the Earth's surface.

Initially, the medium is assumed to be in equilibrium. The equilibrium equations may be obtained by setting a suffix zero with the variables. Next we take the perturbed form of the equations. The variables in the perturbed form are considered to take the form. :

$$(\vartheta,\rho,T) \equiv (\vartheta_0,\rho_0,T_0) + (\vartheta',\rho',T'),$$

where the variables with dashes represent the perturbed part. The perturbed equations must be satisfied by the equilibrium equations. These equations are then linearised and the harmonic character of the variables are satisfied (supposing the variables are proportional to Exp i $(rK_r + \theta K_\theta + ZK_z - \omega t)$, where K_r, K_θ, K_z, are respectively the wave numbers along r, θ and z directions and ω is the wave frequency). After all these steps and suppressing the dashes the eqs. (2.1 to 2.4) may be derived as a set of linear equations with the parameters ϑ_r, ϑ_θ, ϑ_z (*i.e.* the components of V) and ρ/ρ_0 as in another study by the present authors [9].

Now to obtain the dispersion relation for the wave propagation in a non-homogeneous medium with vertical temperature gradient the following assumptions are considered and for non-zero solution the determinant of the co-efficients of the resulting equations is equated to zero.

i) The non-homogeneity and temperature gradient exist along the axis of symmetry only and both density and temperature decrease with increase of altitude *i.e.*

$$\left.\begin{aligned} \frac{\partial \rho_0}{\partial r} = \frac{\partial \rho_0}{\partial \theta} = 0, \quad \frac{\partial \rho_0}{\partial z} = -\rho_0 / H \\ \frac{\partial T_0}{\partial r} = \frac{\partial T_0}{\partial \theta} = 0, \quad \frac{\partial T_0}{\partial z} = -T_0 / H_1 \end{aligned}\right\} \tag{2.5}$$

where H and H_1 are respectively the characteristic length for vertical variation of density and temperature. The parameters with suffix zero represent the initial values.

ii) The wave propagates only due to perturbation in the medium *i.e.*, initially there is no velocity component in any direction *i.e.*,

$$\vartheta_{0r} = \vartheta_{0\theta} = \vartheta_{0z} = 0\,; \tag{2.6}$$

iii) the basic density and temperature are steady *i.e.*

$$\frac{\partial \rho_0}{\partial t} = \frac{\partial T_0}{\partial t} = 0; \tag{2.7}$$

iv) the wave propagation is vertical *i.e.*,

$$K_r = K_\theta / r = 0, \quad K_z \neq 0\ ; \text{ and} \tag{2.8}$$

v) the system is considered with respect to the rotating Earth *i.e.*,

$$\Omega_r = \Omega_\theta = \Omega_z = 0. \tag{2.9}$$

Using the above assumptions, the dispersion relation by which the vertical wave propagation is guided may be obtained as,

$$\frac{\omega^2}{u^2} = \left\{K_z^2 + \frac{1}{\gamma H}\left(\frac{1}{H} - \frac{1}{H_1}\right)\right\} + iK_z\left\{\frac{1}{H} - \frac{1}{\gamma}\left(\frac{1}{H} - \frac{1}{H_1}\right)\right\} \tag{2.10}$$

where, $\gamma = c_p / c_v = 1.4$, the ratio of specific heats

$u = (\gamma R T_0)^{1/2}$, the sound velocity.

3. Analysis of the vertically propagating wave

3.1. Region of positive temperature gradient :

We now suppose that the wave is propagating vertically from a region where the temperature increases (decreases) with increase (decrease) of propagation length *i.e.*, in a medium with positive temperature gradient. Since the dispersion relation (2.10) has been obtained assuming the density and temperature gradient as negative [*vide* (2.5)], for a region with positive temperature gradient H_1 in (2.10) may be replaced by $-H_1$. To study the wave propagation with real frequency (ω) let us write this dispersion relation in the form :

$$K_z = \frac{1}{2}\left[-iq \pm \sqrt{d}\,\right] \tag{3.1}$$

where,
$$q = \frac{1}{H} - \frac{1}{\gamma L} = \frac{1}{\gamma}\left(\frac{\gamma - 1}{H} - \frac{1}{H_1}\right)$$

$$\frac{1}{L} = \frac{1}{H} + \frac{1}{H_1}$$

$$d = \frac{4\omega^2}{u^2} - \left[\frac{1}{H} + \frac{1}{\gamma L}\right]^2.$$

It is noted that here L may be taken as the equivalent characteristic length for density and temperature variation.

Let us consider different possible cases from (3.1) for real ω keeping in mind that the perturbations are proportional to Exp $i\,(ZK_z - \omega t)$.

Case I :

$$q = 0 \;\; i.e\; (\gamma - 1) = H/H_1$$

(a) when $d \geq 0$ *i.e* $\omega^2 \geq \frac{u^2}{H^2}$ (using $q = 0$), we have $K_z \geq 0$. Thus stable harmonic waves or standing waves will be obtained corresponding to the > and = sign respectively.

(b) when $d < 0$ (*i.e* $d = -|d|$) *i.e.* $\omega^2 < \frac{u^2}{H^2}$ (using $q = 0$), K_z is imaginary. Thus damped or amplified waves may be produced corresponding to the damping/amplification factor $\text{Exp}\left(\mp \frac{1}{2}\sqrt{|d|}\right)$

Case II :

$$q > 0 \; i.e.\; (\gamma - 1) > H/H_1$$

(a) when $d \geq 0$ *i.e.* $\omega^2 \geq \frac{u^2}{4}\left[\frac{1}{H} + \frac{1}{\gamma L}\right]^2$, the imaginary part of K_z will be negative and using $q>0$ we obtain $\omega^2 >> \frac{u^2}{\gamma^2 L^2}$ and $\omega^2 > \frac{u^2}{\gamma^2 L^2}$ corresponding to the > and = sign respectively. With these values of K_z the vertically propagating waves will thus undergo amplification with the amplification factor Exp $q/2$.

(b) when $d < 0$ (*i.e.* $d = -|d|$), using $q > 0$ we have $\omega^2 << \frac{u^2}{H^2}$. In such a situation both the terms of K_z are imaginary. Corresponding to the negative sign with $\sqrt{d}$ in (3.1) the waves will undergo amplification with the amplification factor $\text{Exp}\frac{1}{2}\left(q + \sqrt{|d|}\right)$. Corresponding to the positive sign we have,

(i) for $q > \sqrt{|d|}$, the waves will undergo amplification with the amplification factor $\text{Exp}\frac{1}{2}\left(q-\sqrt{|d|}\right)$;

(ii) for $q < \sqrt{|d|}$, the waves will be damped with the damping factor $\text{Exp}\left[-\frac{1}{2}\left(\sqrt{|d|}-q\right)\right]$;

(iii) for $q = \sqrt{|d|}$, we have $K_z = 0$ *i.e.* only standing waves may be produced.

Case III : $q < 0$ *i.e.* $q = -|q|$ and $(\gamma-1) < H/H_1$

(a) when $d \geq 0$ *i.e.* $\omega^2 \geq \frac{u^2}{4}\left[\frac{1}{H}+\frac{1}{\gamma L}\right]^2$, using $q < 0$ we obtain either $\omega^2 >> \frac{u^2}{H^2}$ or $\omega^2 > \frac{u^2}{H^2}$ corresponding to the > and = sign respectively. Also the imaginary part of K_z will be positive. These values of K_z will thus yield damped vertically propagating waves with damping factor $\text{Exp}\left[-|q|/2\right]$.

(b) when $d < 0$ *i.e.* $d = -|d|$ and $\omega^2 < \frac{u^2}{4}\left[\frac{1}{H}+\frac{1}{\gamma L}\right]^2$, both the terms of K_z will be imaginary and we obtain (using $q < 0$) $\omega^2 << \frac{u^2}{\gamma^2 L^2}$. Corresponding to the positive sign with $\sqrt{d}$ in (3.1) the values of K_z will yield damped waves with damping factor $\text{Exp}\left[-\frac{1}{2}\left(|q|+\sqrt{|d|}\right)\right]$. Again corresponding to the negative sign we have,

(i) for $|q| > \sqrt{|d|}$, the waves will undergo damping with damping factor $\text{Exp}\left[-\frac{1}{2}\left(|q|-\sqrt{|d|}\right)\right]$;

(ii) for $|q| < \sqrt{|d|}$, the waves will undergo amplification with amplification factor $\text{Exp}\frac{1}{2}\left(\sqrt{|d|}-|q|\right)$ and

(iii) for $|q| = \sqrt{|d|}$ *i.e.* $K_z = 0$. only standing waves may be produced.

3.2. Region of negative temperature gradient :

We now consider the vertical wave propagation in a region where temperature decreases (increases) with increase (decrease) of propagation length *i.e.* in a medium with negative temperature gradient. It may be recalled that for such a medium the dispersion relation is given by (2.10). To study the wave propagation with real frequency (ω), the relation (2.10) may be arranged in a form similar to (3.1) *i.e.*,

$$K_z = \frac{1}{2}\left[-iq' \pm \sqrt{d'}\right] \tag{3.2}$$

where,

$$\frac{1}{L'} = \frac{1}{H} - \frac{1}{H_1}$$

$$q' = \frac{1}{H} - \frac{1}{\gamma L'} = \frac{1}{\gamma}\left(\frac{\gamma-1}{H} + \frac{1}{H_1}\right) > 0, \qquad \text{for all values of } H \text{ and } H_1.$$

$$d' = \frac{4\omega^2}{u^2} - \left(\frac{1}{H} + \frac{1}{\gamma L'}\right)^2$$

As in § 3.1, let us now consider different possible cases from (3.2) for real ω when $q' > 0$ *i.e.* $\frac{H}{H_1} > (1-\gamma)$.

Case I :

when $d' \geq 0$ *i.e.* $\omega^2 \geq \frac{u^2}{4}\left(\frac{1}{H} + \frac{1}{\gamma L'}\right)^2$,the imaginary part of K_z will be negative and corresponding to the > and=sign respectively we obtain either $\omega^2 >> \frac{u^2}{\gamma^2 L'^2}$ or $\omega^2 > u^2/\gamma^2 L'^2$. These values of K_z will thus yield amplified vertical propagating waves with amplification factor Exp $q'/2$.

Case II :

when $d' < 0$ *i.e.* $d' = -|d'|$ and $\omega^2 < \frac{u^2}{4}\left(\frac{1}{H} + \frac{1}{\gamma L'}\right)^2$, both the terms of K_z will be imaginary and (using $q' > 0$) we obtain $\omega^2 << \frac{u^2}{H^2}$. Corresponding to the negative sign with $\sqrt{d'}$ in (3.2) the values of K_z yield amplified waves with the amplification factor Exp $\frac{1}{2}\left(q' + \sqrt{|d'|}\right)$. Again corresponding to the positive sign we have

(i) for $q' > \sqrt{|d'|}$, the waves will undergo amplification with the amplification factor Exp $\frac{1}{2}\left(q' - \sqrt{|d'|}\right)$;

(ii) for $q' < \sqrt{|d'|}$, the waves may undergo damping with the damping factor Exp $\left[-\frac{1}{2}\left(\sqrt{|d'|} - q'\right)\right]$; and

(iii) for $q' = \sqrt{|d'|}$, *i.e.* $K_z = 0$, only standing waves may be produced.

3.3. *Isothermal region :*

Let us now consider the vertical wave propagation in an isothermal region *i.e.* where H_1 For such a region one may obtain from the dispersion relation (3.2)

$$L' = H, \quad q' = \frac{\gamma - 1}{\gamma H} > 0, \text{ for all values of } H$$

$$d' = \frac{4\omega^2}{u^2} - \frac{1}{H^2}\left(1 + \frac{1}{\gamma}\right)^2.$$

Thus as discussed in § 3.2, the waves while propagating through such a medium will undergo amplification or damping with similar amplification or damping factors.

4. Damping and amplification of waves in the Earth's atmosphere

We now attempt to apply the analytical results of the previous sections for estimating the frequency of the perturbed wave (ω) in damping/amplification in different regions of the Earth's atmosphere. In doing so, we take the sound velocity (u) = 0.33 Kms^{-1} and the characteristic length of density variation (H) = 8 Km., [10] to be approximately constant for the different regions.

Let us first consider the region of positive temperature gradient (*e.g.* stratosphere). For such a region taking the characteristic length of temperature variation (H_1) = 40 Km, we note that $q = 0.1786 > 0$. Thus, on the basis of the analysis in case II of § 3.1 the limits of ω are obtained as follows:

(i) for $d \geq 0$ *i.e.* $\omega \geq 0.0383\ S^{-1}$, the waves always undergo amplification;

(ii) for $d < 0$ *i.e.* $\omega < 0.0383\ S^{-1}$, the waves may undergo amplification corresponding to the negative sign with $\sqrt{d}$ in (3.1);

(iii) for $d<0$ and $q>\sqrt{|d|}$ *i.e.* $\omega > 0.0382\ S^{-1}$, the wave may undergo amplification corresponding to the positive sign with $\sqrt{d}$ in (3.1); and

(iv) for $d < 0$ and $q \leq \sqrt{|d|}$ *i.e.* $\omega \leq 0.0382\ S^{-1}$, the waves may be damped or standing waves may be produced corresponding to the positive sign with $\sqrt{d}$ in (3.1).

Next, for a region of negative temperature gradient (*vide* § 3.2) the limits of ω may be estimated separately for mesosphere and troposphere in view of the variations of H_1 in these regions.

In the mesosphere, we have $H_1 = 40$ Km and $q' = 0.05357 > 0$. The limits of ω are obtained as follows:

(v) when $d' \geq 0$ *i.e.* $\omega \geq 0.0324\ S^{-1}$, the waves always undergo amplification;

(vi) when $d' < 0$ *i.e.* $\omega < 0.0324\ S^{-1}$, the waves may be amplified corresponding to the negative sign with $\sqrt{d'}$ in (3.2);

(vii) when $d' < 0$ and $q' d' \sqrt{|d'|}$ *i.e.* $\omega > 0.0312\ S^{-1}$, the waves may be amplified corresponding to the positive sign with $\sqrt{d'}$ in (3.2); and

(viii) when $d' < 0$ and $q' \leq \sqrt{|d'|}$ *i.e.* $\omega \leq 0.0312\ S^{-1}$, the waves may be damped or standing waves may be produced corresponding to the positive sign with $\sqrt{d'}$ in (3.2).

In the troposphere, we have $H_1 = 11$ Km and $q' = 0.10065 > 0$. The limits of ω are as follows :

(ix) when $d' \geq 0$ *i.e.* $\omega \geq 0.0246\ S^{-1}$ the waves always undergo amplification.

(x) when $d' < 0$ *i.e.* $\omega < 0.0246\ S^{-1}$, the waves may be amplified corresponding to the negative sign with $\sqrt{d'}$ in (3.2);

(xi) when $d' < 0$ and $q' > \sqrt{|d'|}$ *i.e.* $\omega > 0.0182\ S^{-1}$, the waves may be amplified corresponding to the positive sign with $\sqrt{d'}$ in (3.2); and

(xii) when $d' < 0$ and $q' \leq \sqrt{|d'|}$ *i.e.* $\omega \leq 0.0182\ S^{-1}$, the waves may be damped or standing waves may be produced corresponding to the positive sign with $\sqrt{d'}$ in (3.2).

The different regions of the atmosphere, mentioned above are separated by isothermal layers (*e.g.*, stratopause, tropopause etc.). For the isothermal regions ($H_1 \to \infty$) we have $q' = 0.0357 > 0$. The limits of ω on the basis of §3.3 are as follows :

(xiii) when $d' \geq 0$ *i.e.* $\omega \geq 0.0354\ S^{-1}$, the waves always undergo amplification;

(xiv) when $d' < 0$ *i.e.* $\omega < 0.0354\ S^{-1}$ the waves may be amplified corresponding to the negative sign with $\sqrt{d'}$ in (3.2);

(xv) when $d' < 0$ and $q' > \sqrt{|d'|}$ *i.e.* $\omega > 0.0349\ S^{-1}$, the waves may be amplified corresponding to the positive sign with $\sqrt{d'}$ in (3.2); and

(xvi) when $d' < 0$ and $q' \leq \sqrt{|d'|}$ *i.e.* $\omega \leq 0.0349\ S^{-1}$, the waves may be damped or standing waves may be produced corresponding to the positive sign with $\sqrt{d'}$ in (3.2).

Now, considering the different regions of the Earth's atmosphere as an integrated system the above results, obtained in the perspective of the analyses in the previous sections, may conveniently be summarised as in the following table.

Region in the Earth's atmosphere	limit of frequency (ω) in S^{-1}	Nature of waves *
1.Mesosphere	$\omega > 0.031$	A
	$\omega \leq 0.031$	D or S
2.Stratosphere	$\omega > 0.038$	A
	$\omega \leq 0.038$	D or S
3.Troposphere	$\omega > 0.018$	A
	$\omega \leq 0.018$	D or S
4.Isothermal layers	$\omega > 0.035$	A
	$\omega \leq 0.035$	D or S

* A-amplified, D-damped, S-standing

From the above table, it is observed that a vertically propagating wave which is amplified in a region may become damped in another region. Particularly if the wave frequency (ω) lies in the limit $0.031\ S^{-1} < \omega < 0.038\ S^{-1}$, then the wave in the mesosphere (a region of negative temperature gradient) will undergo amplification and in the stratosphere (a region of positive temperature gradient) it will be damped. But during its further downward propagation through the troposphere (a region of negative temperature gradient) the same wave will undergo amplification. However, while propagating through the isothermal layers (*e.g.*, stratopause, tropopause etc.) the wave may be amplified or damped depending on the wave frequency in the aforesaid limit.

Now while being amplified, the wave will absorb thermal energy from the medium and therefore, will cause a decrease in temperature of the medium. Conversely while being damped, the temperature of the medium will increase. (Of course the increase or decrease of temperature will depend on the corresponding damping or amplification factor).

From the above discussion one may thus expect cooling in the mesosphere and troposphere, but warming in the stratosphere when the wave frequency lies in the said limit. If, however, we look into the Earth's lower atmosphere (troposphere) we find that wave amplification is possible for a comparatively wider range of frequency : $0.018\ S^{-1} < \omega < 0.038\ S^{-1}$ (vide table). Since this range is much wider than the former, there may exist waves which are not amplified in the mesosphere but are amplified in the troposphere resulting a in cooling of the medium.

Now to test the validity of the present analysis, one may consider that the perturbations in the Earth's atmosphere increases with the passage of more solar wind during enhanced solar activity. This enhanced perturbation evidently will cause more amplification/damping in different regions. Hence the cooling and warming in the different zones of the Earth's atmosphere, will possibly be more significant at the time of enhanced solar activity. In fact, the statistical analysis [11] also reveals a decreasing trend in the average tropospheric temperature with increase of solar activity, which is consistent with the present analytical result.

5. Concluding remarks

In the foregoing analyses and discussion we have seen that the temperature gradient in the different vertical zones of the Earth's atmosphere are responsible for the damping and amplification of the vertically propagating waves, particularly when the wave frequency lies in a definite range. Such damping and amplification processes, in turn, may be relevant to increase and decrease in temperature in the different zones *i.e.*, troposphere, stratosphere and mesosphere of the Earth's atmosphere.

However, the present analysis has been done considering the wave flow as unidirectional. This study may be extended with non-zero azimuthal and radial wave numbers, the results of which may be more close to the reality. Also it will be interesting to consider the generation mechanism and energetics of the perturbed waves discussed in this paper, which the present authors like to take up in a subsequent study.

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